Quadratic Voting *

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Abstract

We propose *Quadratic Voting* (*QV*) as a method for binary collective decision-making: individuals buy votes for their preferred alternative, paying the square of the number of votes purchased. Quadratic cost uniquely makes the marginal cost proportional to votes purchased, encouraging voting proportional to the value and thus maximizing welfare. Similar heuristic arguments and experiments suggest it is more robust than other efficient mechanisms and it grows naturally from a variety of ideas in disparate literatures. Nonetheless, formal analysis beyond simple examples has proved illusive. By explicitly characterizing the subtle statistical structure of equilibrium, we prove convergence towards optimality in large populations in a canonical environment.

Keywords: social choice, collective decisions, large markets, costly voting, vote trading

*This paper supersedes the authors' previous joint paper "Nash Equilibria for a Quadratic Voting Game", which contained proofs of the main results of this paper, and a previous working paper by Weyl, "Quadratic Vote Buying", which conjectured the form of the Nash equilibria established here. Weyl is co-founder of a commercial venture, Collective Decision Engines, which is commercializing Quadratic Voting for market research and thus has a financial interest in the success of this mechanism. We are grateful to Nageeb Ali, Eric Budish, Bharat Chandar, Jerry Green, Mark Satterthwaite, José Scheinkman and Bruno Strulovici, as well as many other colleagues and seminar participations for helpful comments. We acknowledge he financial support of the National Science Foundation Grant DMS - 1106669 received by Lalley and of the Alfred P. Sloan Foundation, the Institut D'Économie Industrielle and the Social Sciences Division at the University of Chicago received by Weyl. Kevin Qian, Tim Rudnicki, Matt Solomon and Daichi Ueda supplied excellent research assistance. We owe a special debt of gratitude to Lars Hansen, who suggested our collaboration, and to Eric Maskin for an excellent formal discussion of the paper. All errors are our own.

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1 Introduction

> (T)he "one man one vote" rule gives everyone minimum share in public decision-making, but it also sets...a maximum...it does not permit the citizens to register the widely different intensities with which they hold their respective political convictions and opinions.

– Albert O. Hirschman, *Shifting Involvements: Private Interest and Public Action*

A group of N individuals must choose between two alternatives; for example, homeowners may be deciding whether to build a communal swimming pool out of community funds. Each individual has a privately-known value that determines her willingness to pay for the pool. A standard approach would be for individuals to vote on whether to approve the project and for the majority to rule. However, as highlighted by Hirschman above, this rations rather than prices influence on the decision and thus will often fail to maximize dollar-equivalent welfare. For example, a minority of homeowners may disproportionately benefit from the pool, sufficiently that they would be willing to pay off opponents in aggregate to accept it, and yet may be defeated by a simple majority vote.

We propose a simple and detail-free solution: *Quadratic Voting* (*QV*).[1] Individuals buy votes (either negative or positive, depending on which alternative is favored) from a clearing house, paying the square of the number of votes purchased. The sum of all votes purchased then determines the outcome according to a smoothed version of majority rule described more precisely in Subsection 3.1 below. Funds raised are refunded in an essentially arbitrary manner; for example each individual may receive an equal share of payments by all individuals other than herself.[2]

Heuristic analysis suggests both that this mechanism approximately maximizes welfare in large populations and that it is more robust to a variety of concerns (collusion, aggregate uncertainty, non-instrumental voting motivations, the use of artificial currency etc.) than are other efficient mechanisms. Furthermore, building off a now-defunct previous version of this paper (Weyl, 2012), several experimental (Goeree and Zhang, Forthcoming; Cárdenas et al., 2014) and field (Quarfoot et al., Forthcoming; Holland, 2016) studies of QV have found results consistent with these conclusions outside idealized theoretical conditions. However, the formal basis of these results has lagged because even

[1] While we describe the precise provenance of this idea below, as far as we are aware all literature on the topic cites the present paper or previous and now-defunct versions of this paper as the original source of this idea. As such we view QV as a proposal we are making here, rather than as a pre-existing idea which we are studying.

[2] See Park and Rivest (Forthcoming) for a detailed discussion of the costs and benefits of different refund schemes.

in the most standard models, as we show in Section 4, these heuristics are literally false, holding only as an approximation.

This paper makes a first step towards moving beyond these informal and empirical arguments. We prove that in any symmetric Bayes-Nash equilibrium of a private values environment where valuations are drawn independently and identically according to any smooth distribution with bounded support, the welfare loss of QV converges as $N \to \infty$ to a fraction 0 of potential welfare, at a rate $1/N$ for generic value distribution parameters (viz. so long as the mean of the value distribution is not equal to 0). Our analysis is constructive and describes the approximate structure of Bayes-Nash equilibria. We are hopeful that the techniques we develop may be useful in analyzing QV in a broader set of contexts, as suggested by the analysis of Weyl (Forthcoming), but the detailed formalism required to establish our results restricts us here to this rather special setting.

Nonetheless, the heuristic rationale for QV is simple and significantly broader. It seems reasonable to suppose that the benefit a single voter in a large population derives from votes is linear in the number of votes she has and proportional to her value from changing the outcome. If so, her marginal benefit of an additional vote is constant and proportional to her value. For example, if a voter is purely instrumental and rational, her value for a marginal vote is her value for changing the outcome multiplied by her *marginal pivotality*, the amount by which a marginal vote changes the probability of the outcome going in the voter's desired direction. She maximizes utility by equating this marginal utility to the linear marginal cost of a vote. Therefore, if all voters share the same constant of proportionality (e.g. marginal pivotality), they will buy votes in proportion to their values, thus maximizing welfare.

Furthermore, the quadratic cost function is the *unique* cost function with this property. We develop this heuristic justification mathematically, and show how it extends beyond the purely instrumental case and applies also when constants of proportionality may be heterogeneous but independent of individual values, in Subsection 2.1. While this argument is natural, we will show that in the most canonical model it is literally false. Establishing, as we do in Section 5, that it is a sufficiently good approximation with high enough probability to imply approximate welfare maximization turns out to require a subtle and technical statistical argument.

The broad idea of quadratic pricing of public goods dates to the work Groves and Ledyard (1977a), who proposed it as a Nash implementation of the optimal level of continuous public goods under complete information that avoids the fragility, discussed in the next paragraph, of previously suggested efficient mechanisms.[3] In an unpublished

[3]Another common criticism of these mechanisms is that they violate individual rationality constraints

manuscript, Hylland and Zeckhauser (1980, henceforth HZ) provided the first variant of the heuristic rationale above to uniquely justify quadratic pricing mechanism and proposed an iterative procedure that they conjectured would converge to Groves and Ledyard's complete information optimum in the presence of private information. They also showed how quadratic pricing could be applied in environments without private goods by allowing individuals to use an artificial currency to express relative preferences over a collection of public goods to achieve a Pareto-efficient allocation.

Despite this history, Weyl (2012) was the first to propose QV for binary collective decision problems. He conjectured that it would lead to asymptotically efficient decisions in the environment we consider based on the heuristic rationale we present in the next section. This turned out to be an extension of HZ's heuristic reasoning, but he was not aware of this at the time. Goeree and Zhang (Forthcoming, henceforth GZ) independently suggested using a detail-based, approximately direct variant of QV in the special case where values are sampled from zero-mean normal distributions, and derived an equilibrium in the case $N = 2$.[4]

QV is not the first (asymptotically) efficient mechanism economists have proposed for the binary collective-decision problem in the independent private values setting we formalize in Section 3.1 and study in Section 5. The Vickrey (1961)-Clarke (1971)-Groves (1973) (VCG) and "expected externality" (EE; Arrow, 1979; d'Aspremont and Gérard-Varet, 1979) mechanisms are both fully efficient even in finite populations and the "implement-the-mean preference" (IM; Krishna and Morgan, 2001) and "costly voting" (CV; Ledyard, 1984) mechanisms are asymptotically efficient (generically). There have been concerns in the literature, however, about the practical relevance of these welfare results outside of the models in which they are established. The EE and IM mechanism require the mechanism designer to know the value distribution and are not defined in cases in which there is aggregate uncertainty about this value distribution. The VCG mechanism is extremely sensitive to collusion, in the sense that any two individuals have a collusive *equilibrium* in which they obtain their first-best payoff and does not naturally extend to environments

if individuals are entitled to the status quo outcome. Mailath and Postlewaite (1990) show that, in this environment and if entitlements to one alternative (call it the status quo) is provided to all agents, then as the population grows large as long as any event exists in which the status quo is optimal, the probability of the alternative being adopted tends toward 0. This weakness applies equally to QV as to any of the other mechanisms for this setting. As a result we consider an environment in which the alternatives are symmetric and no individual has an entitlement to any alternative. In this environment individual rationality does not arise as a constraint.

[4]They also informally characterize equilibrium for sufficiently large N in the general case when the distribution has a mean of zero, but this relies on assuming that limiting approximations (such as the central limit theorem) hold exactly, which they do not for any distribution we are aware of as we discuss in detail in Section 4.

where monetary transfers are undesirable. The CV mechanism is not efficient if individuals deviate from perfect instrumentality and rationality in ways that would, for example, explain the relatively high levels of turnout observed under present voting systems.

In contrast, recent laboratory (GZ), small-scale field (Cárdenas et al., 2014) and large-scale field (Quarfoot et al., Forthcoming; Holland, 2016) experiments suggest that QV achieves high level of welfare well outside the ideal conditions we study, including in cases where individuals play clearly at odds with the predictions of Bayes-Nash equilibrium, and extends naturally to settings where monetary transfers are unavailable or viewed as unfair (Posner and Stephanopoulos, Forthcoming). Furthermore numerical simulations and asymptotic approximations by Weyl (2015) suggest QV usually performs within a few percentage points of full optimality in a variety of finite population sizes and analysis based on analytic approximations by Weyl (Forthcoming) suggests that QV is significantly more robust to collusion, aggregate uncertainty and deviations from perfect instrumental rationality than are existing mechanisms.[5]. Despite this, to our knowledge none of these results provides a rigorous proof of (asymptotic) optimality in the sort of non-cooperative, incomplete information game theoretic model in which mechanisms for allocating private goods have been studied at least since the work of Myerson (1981). The primary contribution of the present paper is to provide the first such rigorous result and tools that may be useful in formalizing other approximation and heuristic arguments.

Most previous results on convergence of mechanisms (for private goods) towards optimality show that agents' strategies converge to a simple, determinate continuum equilibrium strategy, typically involving truthful direct demand revelation.[6] However, because marginal pivotalities converge to zero in the binary collective decision problem, there is no well-defined (finite) continuum analog. Previous asymptotic optimality analysis of voting (Ledyard, 1984; Feddersen and Pesendorfer, 1997; Myerson, 2000; Krishna and Morgan, 2015) focuses on environments where individuals have at most three choices (vote for either alternative or abstain). In such settings individuals' marginal pivotalities are mechanically equal by the assumption of independent and identical sampling and a symmetric equilibrium.

In QV, on the other hand, individual choices lie in a continuum and thus different individuals will typically have different marginal pivotalities because they purchase different numbers of votes and thus move the chance of a tie through these votes by different amounts. The central technical challenge in our results is proving that in equilib-

[5]QV may therefore be motivated by the recent literatures on robust mechanism (Wilson, 1987; Bergemann and Morris, 2005) and market (Roth, 2002) design that seeks simple, robust and approximately efficient, rather than fully optimal, mechanisms

[6]For example see Roberts and Postlewaite (1976), Rustichini et al. (1994), and Che and Kojima (2010).

rium these variations in strategies do not cause marginal pivotalities to differ by much for most individuals. While the theorems of Kahn et al. (1988) and Al-Najjar and Smorodinsky (2000) imply marginal pivotality must converge to zero as $N \to \infty$, we show that nonetheless with probability approaching 1, the ratio between the marginal pivotalities of two randomly chosen voters will be close to 1. Thus, although the heuristic rationale for QV is similar to that put forward by Ledyard (with the uniform distribution of voting costs independent of values he considers, the cost of a turnout is effectively quadratic in the aggregate), the equilibrium construction with which we are primarily concerned here bears little relationship to that in CV models.

In fact, as we will show, when the value distribution has non-zero mean equilibria take a particularly exotic form, with nearly all individuals buying vanishingly few votes but a vanishingly small number of "extremists" buying a relatively large number, enough to single-handedly sway the election. Large$-N$ optimality emerges because the expected number of extremists in the population decays as $1/N$. In contrast, when the mean value is zero, such extremists will not exist. Here optimality will depend on another subtle property of our equilibrium characterization. Specifically we show that population variability of marginal pivotalities is sufficiently small that deviations in the vote total stemming from nonlinearities of the equilibrium voting strategy vanish compared to variations arising from the sampling variability of the agents' values.

Given the technical nature of our proofs, we provide all of these in an online appendix available at http://ssrn.com/abstract=2790624 and in the text focus on statements of our results, along with detailed outlines of our proofs. Our paper is divided into roughly two parts that can be read to some extent separately. Section 2 cover the economic background, motivation and broad intuitions surrounding and arguments for QV. Section 3 defines our formal model. Section 4 describes why the intuitions developed in Section 2 do not immediately imply efficiency in large populations, motivating our formal analysis in Section 5 and our online appendix.

Before turning to those tasks, we briefly discuss how we imagine QV being of practical value (or not). The version of QV we study formally in this paper uses monetary transfers to achieve aggregate surplus maximization. This is a common criterion of interest in economics and is also maximized by other previous solutions such as VCG, EE, IM and CV. However, in the context of collective decisions it has been widely criticized as unjustly under-weighting the interests of the poor or otherwise budget-constrained. As Laurence and Sher (Forthcoming) and Ober (Forthcoming) argue, this may make the form of QV we analyze here unattractive for consequential public decisions, though see Posner and Weyl (2015, Forthcoming) for a contrasting view. Nonetheless, the version of QV we analyze here may still be desirable in contexts where equity and budget constraints are significant concerns such as debt settlements (Posner and Weyl, 2013), corporate governance (Posner and Weyl, 2014) or cooperative real estate management.

Furthermore, QV has been adapted by Posner and Stephanopoulos (Forthcoming) and Quarfoot et al. (Forthcoming) to settings where an artificial currency is used to trade off across various referenda, a solution that Laurence and Sher and Ober argue overcomes the leading ethical objections to QV but to which the relevance of our results is speculative. Investigating whether our results can be extended to prove approximate Pareto efficiency in those settings is therefore an important direction for future research.

2 Motivation

The next two sections provide techniques for formalizing the analysis of QV. However, such analysis is only worth the substantial trouble required if QV is a potentially useful mechanism for collective decision-making. In this section we make the case for the promise of QV based on a number of attractive properties that do not require detailed formal analysis to verify.

2.1 Heuristic rationale

Suppose there are two alternatives 1 and 0 and each individual $i \in 1, \ldots N$ is willing to pay $2u_i$ to see alternative 1 adopted rather than alternative 0; if u_i is negative the individual prefers alternative 0. Each individual i must choose a number of votes $v_i \in \mathbb{R}$ to buy on the choice at a cost of v_i^2 with the interpretation that $v_i > 0$ constitutes a vote for alternative 1 and $v_i < 0$ constitutes a vote for alternative 0. The alternative 1 is implemented with a probability monotonically increasing in the aggregate votes $V \equiv \sum_i v_i$ and the alternative 0 with the complementary probability. We defer detailed discussion of this mapping until Subsection 3.1 below, but for now think of it as majority rule where if V is positive alternative 1 is implemented and if V is negative alternative 0 is implemented.

The value individuals derive from the votes they have is a long subject of debate in political economy and the simplest model of individuals as perfectly rational and instrumental has weak empirical support (Blais, 2000). However, in a large society, most individuals make a small contribution to the vote total. This suggests that individuals' value for votes lis likely to be approximately linear in the number of votes she casts so long as her utility is driven by the impact she has on the vote total, as argued by Mueller (1973, 1977) and Laine (1977). Furthermore this value should be proportional to the value the voter has for changing the outcome: the more important the outcome is to the individual, the more she should value her votes.

If the voter-specific constant of proportionality in this relationship is ϵ_i, then voter i's gain from votes is $2\epsilon_i u_i v_i$ in units of the numeraire. ϵ_i may represent many factors: the chance that individual i is pivotal (Downs, 1957), the chance individual i incorrectly believes she is pivotal (Uhlaner, 1993), the voter's gain from influencing the vote total to determine the strength of the mandate received by her desired alternative (Schwartz, 1987) or the voter's desire to express her value for its own sake through her vote (Fiorina, 1976). In any of these cases, the representation supposed here is accurate so long as the individual's value does not diminish or increase on the margin as she acquires more votes.

The marginal cost of a vote is $2v_i$. An optimizing individual will equate marginal benefit ($2\epsilon_i u_i$) and marginal cost and therefore optimally purchase a number of votes

$$v_i^\star = \epsilon_i u_i.$$

If each individual behaves this way, the likely winner of the election will be determined by the sign of $\sum_i \epsilon_i u_i$. If all ϵ_i are the same across individuals (and positive), this evidently has the same sign as $\sum_i u_i$, implying that the utilitarian optimal winner is selected. However, even if ϵ_i differs across individuals, optimality will still arise when the population is large so long as u_i and ϵ_i are drawn independently and the mean μ of the distribution of u_i is non-zero. In this case $1/N \sum_i \epsilon_i u_i$ is, with high probability, close to its mean $E[\epsilon]E[u]$, which has the same sign as $E[u]$ (the utilitarian optimal choice) so long as $E[\epsilon] > 0$ (individuals vote in favor of their preferred alternative).

Furthermore this conclusion holds only for the quadratic functional form of the cost. If instead the cost of votes were votes raised to power x (viz. the cost of votes were $2/x v^x$), the marginal cost of a vote would be $2v_i^{x-1}$ and thus it would be optimal for voters to buy votes proportional to $u_i^{1/x-1}$ not to u_i.[7] For example if $x = 4$ rather than 2, optimal votes would be proportional to $\sqrt[3]{u_i}$. In some cases $E\left[\sqrt[3]{u_i}\right]$ might happen to have the same sign as $E\left[u_i\right]$, but in others it will also have the opposite sign. Returning to the pool example from the introduction, if alternative 1 is favored by a minority with strong preferences and opposed by a majority with weak preferences, the quartic rule might lead to tyranny of the majority by under-weighting the intensity of strong preferences.

Thus while the bulk of our analysis below focuses on a much more specific case than the one outlined here, the basic intuition for why QV might be attractive is in our view not driven primarily by the details of the model we consider, but by the broader logic

[7] More formally by $u_i^{1/x-1}$ we mean sign $(u_i)\,|u_i|^{1/x-1}$.

discussed here. However, as we will see, this heuristic logic is imprecise and will turn out to be literally false in its details in most reasonable models. Our analysis will consider the extent to which it is nonetheless a sufficiently precise approximation for its central conclusion to hold approximately in large populations.

2.2 Weaknesses of other efficient mechanisms

Before turning to this analysis, however, it is worth considering whether it is worth the trouble of verifying these approximations. After all, economists have proposed many other mechanisms that achieve optimality in the model we describe formally below, at least as the population size grows large. Is there any reason to suppose QV is superior to these alternatives? We now review weaknesses of existing efficient mechanisms and informally describe properties of QV that suggest it be less fragile than alternatives.

Because so many mechanisms have been proposed and because so many practical issues have been raised with each, we cannot briefly describe them all or all the challenges faced by each. Instead we have discuss four that colleagues have most frequently raised with us, those mentioned in the introduction (VCG, EE, IM and CV) and the concerns with them that seem most serious to us from our reading of the literature.

1. In the VCG mechanism, individuals report values and the decision is determined by the sign of the sum of values. If an individual is *pivotal* (viz. had expressed indifference, the outcome would have gone in the opposite direction) she is forced to pay the amount by which the outcome would have gone in the opposite direction (viz. the minimum she had to report to change the outcome). VCG is efficient even in finite populations under quite general conditions so long as individuals play non-cooperatively the weakly dominant strategy of truthfully revealing their values.

   However, in addition to this efficient equilibrium, there exists, for any two individuals who have the same directional preference (or any individual who can fraudulently represent herself as two individuals), another equilibrium in which those two get their desired outcome and make no payments. Suppose each reports an enormously large value for her favored alternative, enough to easily be greater than the total value of all others combined. This ensures that neither of the two colluders is pivotal and thus neither makes a payment. Such collusive attacks on VCG are pervasive in anecdotal accounts of using it for voting, as well as in the experimental literature (Attiyeh et al., 2000). This and other concerns led Vickrey (1961) and Groves and Ledyard (1977b) to dismiss the practical relevance of VCG and focus on inefficient mechanisms, such as one-person-one-vote, that are more robust to collusion (Bierbrauer and Hellwig, 2016).[8]

Of course, QV is not immune to collusion and fraud. In particular the convex cost function it imposes creates a natural incentive for individuals to try to spread out vote expenditures to avoid the convexity. However, such manipulations have a more gradual effect than under VCG, where just a single act of fraudulent duplication can achieve any outcome an individual wishes at no cost. To see this, consider an individual spending \$100 on votes and thus obtaining 10. If she can instead represent herself as two individuals she can obtain more votes: $2\sqrt{50} \approx 14$. Large scale collusion or fraud could thus potentially cause a problem, as it can as well in standard market economies and under one-person-one-vote rules, but it seems unlikely that very small collusive or fraudulent groups would have a severe impact. Larger scale collusion and fraud may then be deterred by legal prohibitions and enforcement like those against vote buying, voter fraud and cartels.

Another weakness of the VCG mechanism concerns its heavy reliance on monetary transfers. Groves and Ledyard (1977b) argue that VCG has no natural analog in a setting without unlimited and budget-unconstrained monetary transfers, which motivated their use of quadratic pricing as an alternative (Groves and Ledyard, 1977a). For tractability our formal analysis below focuses on the case with monetary transfers. However, the QV mechanism can easily be adapted to a setting in which such transfers are absent by endowing individuals with artificial currency that they can spread across multiple referenda as they see fit. This is analogous to the HZ application of the Groves and Ledyard (1977a) mechanism to a setting without transfers and with multiple continuous public goods and thus while such an adaptation sacrifices global welfare optimality, HZ's arguments suggest it will achieve Pareto efficiency among the set of collective decisions included in the mechanism. In most present applications (Posner and Stephanopoulos, Forthcoming; Quarfoot et al., Forthcoming; Holland, 2016) it is this version of QV that is applied.

2. In the EE and IM mechanisms, the administrator is assumed to know the distribution from which values are independently drawn. In EE, she uses this to calculate the expected VCG payment (over all other individuals' value draws) that the individual will make given the value she reports and charges these to individuals. In IM she simply implements 1 if $\mu > 0$, 0 if $\mu < 0$ and flips a coin if $\mu = 0$. EE achieves op-

[8] More recently and with more detailed formal critiques Ausubel and Milgrom (2005) called it "lovely, but lonely" and Rothkopf (2007) concludes it is "not practical".

timality in a truthful Bayes-Nash equilibrium with any population size, for the same reason as VCG. IM is somewhat simpler but only achieves approximate optimality in large populations, and only in the generic case when $\mu \neq 0$.

However, both EE and IM require the administrator to know the value distribution, a condition that Wilson (1987) argues is often unreasonable. If instead this distribution is uncertain (if there is aggregate uncertainty) then both mechanisms are not even defined.[9] By contrast, QV is defined independent of details of the value distribution and thus may be applied in environments with aggregate uncertainty about that distribution. Such uncertainty may cause inefficiencies for QV; our analysis implying full asymptotic optimality below requires all individuals have values draw independently and identically from a commonly known value distribution. If this fails, ϵ_i and u_i will be correlated and this may cause welfare loss. However, QV, unlike EE and IM, may still be used in such cases.

3. In CV, individuals may only make one of three choices: vote in either direction or abstain. However it is assumed that voting is costly. If a) voting is strictly costly for all individuals, b) voting costs are drawn from a continuous probability density function with positive density at 0 and c) voting costs are drawn independently of values, Ledyard (1984) shows that optimality is achieved in large populations.[10] In large populations, a vanishing fraction of the population votes as the chance of being pivotal declines. The fraction of individuals with a given value voting can thus substitute for the role played by the number of votes an individual casts in QV. Any cost distribution with positive density at 0 is approximately uniform about 0 and thus the total cost of votes is approximately quadratic in the number of individuals with that value voting in large populations. Thus CV, under these conditions, yields roughly the same results as QV, yielding large population optimality.

   However, as Ledyard argues this result is of limited practical relevance because a) observed turnout rates dwarf those required for this argument and b) there are many reasons to believe, and much empirical evidence suggesting, that revealed voting costs are correlated with preferences. If individuals have motives for voting that do not vanish as the population size grows large, the shape of the distribution of voting costs that matter will not be uniform and thus there is no guarantee of

[9] Similar concerns apply to efficient threshold voting schemes proposed by Ledyard and Palfrey (1994, 2002); while the first paper allows for a common values aggregate uncertainty, neither allow for aggregate uncertainty in private values. EE also suffers similar collusion concerns to VCG.

[10] See also more recent analysis by Myerson (2000) and Krishna and Morgan (2015).

optimality.[11] In contrast as our discussion of the previous subsection emphasized, the large population welfare of QV depends only on the constants of proportionality between votes and values being independent of values and not on the precise details of equilibrium play, such as individuals voting vanishingly little. Thus while CV seems is less fragile and thus likely more practical than the other mechanisms described above, the arguments for its (near) optimality apply less broadly than do those for QV.

Thus it seems plausible that QV may be more robust than existing efficient mechanisms to conditions beyond the simple formal environment we consider below.

2.3 Related literature

Of course these few issues are far from exhausting the broad range of robustness challenge any mechanism would have to confront in order to be a practical proposal. No brief analysis, however, informal, could hope to address all such problems. This discussion can at best be seen as a sampling of potential questions that may arise. Thus we see the most important, though hardest to formalize, feature of QV being its "simplicity".

In particular a possible explanation of the fragility of the mechanisms of the previous subsection to small changes in the modeling environment is the fact that they seem complex and fitted tightly to the particular environment that motivated them. On the other hand, QV strikes us as simpler and more natural than the mechanisms above both because it closely resembles voting rules described in the political economy and social choice literatures and because it can be "derived from" from several different and conflicting approaches in the mechanism design literature. However, given the inherent subjectivity the reader must judge for herself whether she finds QV to be a simpler mechanism than alternatives in the literature on mechanism design. We thus now discuss these literatures to help the reader to reach her own conclusion.

2.3.1 Mechanism design

One way of arriving at QV from the mechanism design literature follows on VCG and especially EE. GZ show that when (non-generically) the value distribution has a mean of 0, EE payments are approximately quadratic in large populations. However, in the case

[11] Correlation between voting costs and values have even more dramatic effects (Kaplow and Kominers, Forthcoming). Thus while CV may be more efficient than universal turnout under fairly broad conditions (Borgers, 2004), it is unlikely to be utilitarian optimal.

when $\mu \neq 0$ EE payments are not close to quadratic in large populations. Thus QV may be seen inspired by the limit of EE, though it is not equal to it typically.

Even more-closely connected is the "Quadratic Mechanism" proposed by Groves and Ledyard (1977a) to address the weaknesses Groves and Ledyard (1977b) saw in VCG. They consider a complete information general equilibrium economy with a vector of continuous public goods and individuals with convex preferences. They proposed individuals simultaneously name their desired level of the public good, paying the square of the amount by which their desired level differs from that requested by others in aggregate. Given complete information, there is a Nash equilibrium in which all individuals request the jointly optimal public good vector for reasons analogous to Subsection 2.1 above.

However, Greenberg et al. (1977) argued that in most practical settings private information is a key constraint and thus the Groves and Ledyard process is unlikely to find the optimum and Maskin (1999) showed that under complete information many other mechanisms can implement the optimum. To address this issue, in a still-unpublished manuscript HZ proposed using a tâttonement-like process to iteratively converge to the Groves and Ledyard optimum, in a model with only public goods where an artificial currency is used to constrain requested changes to the public good bundle. HZ proposed a version of the heuristic argument above to suggest that a quadratic rule is uniquely incentive compatible at this optimal point. However, they did not consider strategic incentives during the tâttonement process converging to this point. For private goods markets, Roberts and Postlewaite (1976) show that it is during tâttonement that individuals have an incentive to exercise market power. As such, despite their game theoretic formulation, HZ's focus is on a limiting "price taking" model.

QV and our analysis of it differ from HZ in three ways.[12] First, HZ consider a general equilibrium economy without external transfers, while we assume quasi-linear utility. Second, HZ assume continuous goods with strictly concave preferences so that the optimal allocation lies on the interior, while we consider a binary public good with linear preference so either full implementation or full non-implementation is generically optimal. We use Hylland and Zeckhauser's (1979) insight that probability shares ("votes") may be used as a substitute for continuous goods to extend logic from continuous cases to discrete ones. Finally, we are concerned with the convergence of finite population strategic incentives to the perfectly competitive limit, rather than with the limit itself.[13]

[12] Interest in the HZ mechanism has recently revived. Benjamin et al. (2013) propose and Benjamin et al. (2014) axiomatize the HZ mechanism as a way to aggregate subjective well-being data into public policy. Chung and Duggan (2015) propose it as an equilibrium concept for spatial policy bargaining models. None of these consider strategic issues, allow a numeraire good or discrete choices, and thus they are strictly more distantly related to our work than is that of HZ.

[13] In all of these ways, the relationship between our work and that of HZ, as well as Groves and Ledyard

Nonetheless HZ clearly shares many structural and motivational similarities with QV. See Tideman and Plassmann (Forthcoming) for a more detailed discussion of the relationship of QV to the mechanism design literature and Benjamin et al. (Forthcoming) for a more formal discussion of the relationship between HZ and QV.

2.3.2 Social choice and voting

While social choice and voting theory has largely focused on mechanisms that reveal only ordinal preferences, recent work proposes protocols that account for cardinal preferences building off a common voting scheme for the election of corporate boards and in some political settings: "cumulative voting".[14] Cumulative voting allocates a collection of votes to each voter to distribute across candidates and proposals. Casella (2005, 2012) proposed allowing votes on binary issues to be "stored" across different issues over time and Hortala-Vallve (2012) proposed allowing them to be used across different binary issues at the same time.

These schemes share with QV the aim of incorporating preference intensity through trade-offs. However, they differs in two respects. First, QV uses a quasi-linear numeraire rather than internal trade-offs. However, given HZ's similar use of an artificial currency, a second difference seems more essential: the budget in these schemes is linear (each vote on an issue is equally costly out of the budget, rather than increasingly costly out of the budget). Because of this, Mueller (1973, 1977) and Laine (1977) argue and Shen (2013) proves that in large populations it will typically be optimal under in these schemes for each voter to put all votes on a single issue. Thus these linear voting schemes represents only the most intense individuals and does not maximize welfare. QV may be seen as an adjustment of the functional form aimed at maximizing welfare.[15]

(1977a), closely parallels the relationship between work on the convergence of the double auction (Satterthwaite and Williams, 1989; Rustichini et al., 1994; Cripps and Swinkels, 2006) and the classical theory of general equilibrium with no external numeraire, continuous goods and price taking behavior.

[14]Another approach that has been less discussed within economics is Smith (2000)'s Range Voting where individuals to express any vote between positive and negative one to express preference intensity. The lack of interest in this system may be because Shen (2013) showed that for strategic individuals in a binary this system does not differ from standard voting as individuals generically choose to use the ends of the range. In the multi-candidate context this "range voting" becomes approval voting.

[15]Jackson and Sonnenschein (2007) proposed a mechanism more tightly aimed at welfare maximization that is too intricate for us to do justice to here, but it roughly involves individuals being given a limited number of times they can claim to have each value type. This evidently requires knowledge by the planner of the value distribution in this game, making it more closely related to the EE and IM mechanisms than to QV.

2.3.3 Voting trading and political economy

QV may also be seen as a form of vote trading.[16] Dekel et al. (2008, 2009) and Dekel and Wolinsky (2012) study models of vote buying in both politics and contests for corporate control using game theoretic analysis; Casella et al. (2012) propose a notion of perfect competition. All generally reach the conclusion that with linear vote trading decisions tend to be dominated by the most intense or a few of the most intense individuals, with other preferences being irrelevant. Again this is consistent with our heuristic logic above as when the power of the cost of votes is x individuals should buy votes roughly proportional to $u^{1/x-1}$. As $x \to 1$ this approaches u^{∞}, making the votes of the most intense individuals infinitely larger than those of any other individual. Thus $x \to 1$ yields the dictatorship of the most intense individual while $x \to \infty$ yields votes proportional to u^0 or one-person-one vote. In this sense linear vote trading (or cumulative voting) is just as far away from QV as one-person-one-vote is. Nonetheless, from another point of view, QV can be seen as a change in the functional form of vote trading designed to ensure optimality.

We thus view QV as something of a bridge between the mechanism design literature, which seeks optimality, and the social choice and political economy literatures, which focus on "simpler" schemes that seem more "practical". QV is similar in its simplicity to vote trading and cumulative voting, but is designed to be utilitarian efficient where these other schemes do not aim to maximize welfare. Its close relation to a variety of generally disjoint literatures suggests its naturalness and that the failure of previous literature to consider it arises primarily from the technical challenges involved in analyzing it, to which we now turn.

3 Preliminaries

We now define the formal environment in which we prove our results.

3.1 Model

We consider an independent symmetric private values environment with N voters $i = 1, \ldots, N$. Each voter i is characterized by a value, u_i; these values are drawn independently and identically from a continuous probability distribution F supported by a finite interval $[\underline{u}, \overline{u}]$, with associated density f and $\underline{u} < 0 < \overline{u}$. For normalization, we assume

[16] Weyl (2012) referred to QV as "Quadratic Vote Buying".

the numeraire has been scaled so that $\min(|\underline{u}|, \overline{u}) \geq 1$. We denote by μ, σ^2, and μ_3, respectively, respectively the mean, variance, and raw third moments of u under F, and we assume f is smooth and bounded away from 0 on $[\underline{u}, \overline{u}]$. The two alternatives are 0 and 1. Individuals are risk-neutral, quasi-linear expected utility maximizers who gain $2u_i$ dollars of utility if option 1 is adopted rather than option 0.

We consider a variant on QV where

1. Each individual chooses a number of votes $v_i \in \mathbb{R}$ to buy.
2. Each pays v_i^2 dollars and receives a refund of $\frac{\sum_{j \neq i} v_j^2}{N-1}$ dollars.[17]
3. The outcome is option 1 with probability $\frac{\Psi(V)+1}{2}$ where $V \equiv \sum_i v_i$ and $\Psi : \mathbb{R} \to [-1, 1]$ is an odd, nondecreasing, C^∞ function[18] such that
   (a) for some $0 < \delta < \infty$,
   $$\Psi(x) = \text{sgn}(x) \quad \text{for all } \; |x| \geq \delta;$$
   (b) Ψ has positive derivative[19] ψ, on the interval $(-\delta, \delta)$; and
   (c) $\psi'(x) > 0$ for all $-\delta < x < 0$ and ψ has a single point of inflection in $(-\delta, 0)$.

We shall refer to Ψ as the *payoff function*, because it determines the quantity by which values u_i are multiplied to obtain the allocative (as opposed to transfer) component of each individual's utility. Arguably the most natural form of Ψ is a discontinuous jump from -1 to 1 at 0 and this is the form it must take in the limit as $\delta \to 0$. However, for technical reasons, we require the smoothness created by a $\delta > 0$ to prove our results. We conjecture that they extend to the limiting case as well and, given this, do not discuss the choice or exact form of Ψ in great detail as our goal is primarily to model this limiting case. Nonetheless, in some examples it may be more natural or realistic to implement a $\delta > 0$. For example, in many practical voting settings it is unrealistic to track all votes with perfect precision, implying that, given votes actually cast it may be necessary to accept some

[17]This particular refund rule is for concreteness and budget balance only, and plays no role in our results. A wide range of refund rules would work equally well; see Park and Rivest (Forthcoming) for a detailed formal discussion.

[18]The payoff function Ψ smooths QV so that close decisions are decided probabilistically, rather than deterministically. While we model it as a feature of the mechanism, it can also be viewed as a product of noise in the environment, possibly arising from a small number of exogenous noise voters or a small, fixed error in the vote-tallying process.

[19]Thus, $\psi/2$ is an even, C^∞ probability density with support $[-\delta, \delta]$ that is strictly positive on $(-\delta, \delta)$. There are infinitely many such probability densities. For example, if $X_1, X_2, \ldots$ is a sequence of independent, identically distributed random variables each with the uniform density on $[-\delta, \delta]$, then the random variable $Y = \sum_{n=1}^{\infty} X_n/2^n$ has a C^∞ density that meets the requirements (a), (b), (c) in condition 3.

uncertainty about the final out come in close elections. In other cases a non-zero δ may be positively desirable, such as when a vote or survey is taken as advisory information that increases the chance of the alternative being implemented but other stochastic factors reviewed by a central authority after the vote will also help determine the outcome.

In most of these cases, Ψ functions satisfying our properties are natural, though not universal. Symmetry of ψ about the origin embeds fairness across the two alternatives. Certainty in the outcome beyond a finite δ is natural in many settings because a strong enough vote is typically decisive. Smoothness of ψ is a common function of probability densities characterizing many forms of uncertainty.

Conditional on the values of $\{v_i\}$, each individual i earns expected utility

$$u_i\Psi(V) - v_i^2 + \frac{1}{N-1}\sum_{j\neq i} v_j^2. \tag{1}$$

Because the last term in this expression is independent of individual i's actions, we will henceforth neglect it. Thus, in a symmetric equilibrium, a voter with value u will maximize[20]

$$\mathbb{E}\left[u\Psi\left(V_{-1}+v\right)\right] - v^2, \tag{2}$$

where $V_{-1} \equiv \sum_{i\neq 1} v_i$ is the *one-out vote total*, the sum of all votes cast by all but a single individual.

To connect this to our informal logic in Subsection 2.1, we differentiate the expression (2) with respect to v yields the following first-order condition for maximization:

$$u\mathbb{E}\left[\psi\left(V_{-1}+v\right)\right] = 2v \implies v(u) = \underbrace{\frac{\mathbb{E}\left[\psi\left(V_{-1}+v(u)\right)\right]}{2}}_{\text{marginal pivotality}} u. \tag{3}$$

The marginal benefit of an additional unit of vote is thus twice the individual's value multiplied by the influence this extra vote has on the chance the alternative is adopted, the vote's *marginal pivotality* discussed in the introduction. The marginal cost of a vote is twice the number of votes already purchased. Thus in this model the conditions of Subsection 2.1 will be satisfied to the extent that this marginal pivotality is, in equilibrium, sufficiently independent of the value u_i a voter draws. Our central results in the next section are concerned with establishing this approximate independence and deriving its consequences for asymptotic optimality.

We define the *expected welfare loss*[21] as $WL \equiv \frac{1}{2} - \frac{\mathbb{E}[U\Psi(V)]}{2\mathbb{E}[|U|]} \in [0,1]$, where $U \equiv \sum_i u_i$.

[20] In our online appendix we will prove that Bayes-Nash equilibria are in essentially pure strategies.

[21] Given our assumption of quasi-linear preferences, utilitarian welfare is equivalent to efficiency in the

This measure is the unique negative monotone linear transformation of aggregate welfare realized $U\Psi(V)$ that is normalized to have range of the unit interval.[22]

3.2 Existence of Equilibria

Lemma 1. For any $N > 1$ a monotone increasing, type-symmetric Bayes-Nash Equilibrium v exists.

This result follows directly from Reny (2011)'s Theorem 4.5 for symmetric games.[23] We focus on symmetric equilibria because, in large symmetric games, asymmetric equilibria typically differ little from symmetric equilibria and are harder to characterize in detail (Satterthwaite and Williams, 1989; Rustichini et al., 1994; Cripps and Swinkels, 2006).[24]

4 Why Heuristics Do Not Suffice

Absent further perspective, the modeling assumptions we make in the previous section (e.g. that $\delta > 0$ and that f has bounded support) and the analysis we provide below may appear unsatisfying. The former are perhaps not the most natural or realistic assumptions we could make and the latter requires technical intricacies that may seem overkill to establish a result with the clear intuition provided in Subsection 2.1 above. We therefore illustrate in this section why these heuristic arguments are insufficient to establish results on asymptotic optimality formally.

To do so, consider a case that is only covered in the limit of our assumptions of the previous section, but is particularly simple to analyze: every individual's value is drawn independently and identically from a normal distribution with mean μ and unit variance and $\delta = 0$ so alternative 1 is implemented if $V \geq 0$ and 0 if $V < 0$. In this case, and assuming that V_{-1} has a smooth density g, the first-order condition in Equation 3 becomes

$$v(u) = 2g\left(-v(u)\right)u. \tag{4}$$

sense of Kaldor (1939) and Hicks (1939) and thus we use the term efficiency interchangeably with welfare.

[22] It is also the complement of the ratio of the gap between the expected realized aggregate utility and the worst possible utility that could be achieved to the gap between the expected first-best utility and the worst possible utility that could be achieved.

[23] All of Reny's conditions can easily be checked, so we highlight only the less obvious ones. Continuity of payoffs in actions follows from the smoothed payoffs imposed through Ψ. Type-conditional utility is only bounded from above, not below, but boundedness from below can easily be restored by simply deleting for each value type u votes of magnitude greater $\sqrt{2\,|u|}$. The existence of a monotone best-response follows from the clear super-modularity of payoffs in value and votes.

[24] However, for this reason, we conjecture that our results extend to all equilibria.

A natural conjecture is that our heuristic rationale from Subsection 2.1 holds exactly for sufficiently large N: every individual would choose $v(u) = \epsilon u$ for some common $\epsilon > 0$. GZ's Proposition 2 informally state this conjecture in the case when $\mu = 0$.[25] As a formal matter, however, this is false, as we now show by contradiction.

Suppose this were true. Then v_i would have a normal distribution with mean 0 and variance $4\epsilon^2$. Thus the distribution of V_{-1} would be normal with mean 0 and variance $4(N-1)\epsilon^2$. Thus by Equation 4 $v(u)$ would have to satisfy, by the definition of the normal density function,

$$\epsilon u = v(u) = \frac{e^{-\frac{v(u)^2}{8(N-1)\epsilon^2}}}{\sqrt{2\pi(N-1)}\epsilon} u = \frac{e^{-\frac{u^2}{8(N-1)}}}{\sqrt{2\pi(N-1)}\epsilon} u,$$

which cannot hold as the right expression is not linear in u. Thus it cannot be the case that $v(u) = \epsilon u$ for any value of ϵ or any value of N. However, for large N it *is* plausible that this relation may hold *approximately* as suggested by the fact that $e^{-\frac{u^2}{8(N-1)}}$ is approximately constant for large N. While we will show that when our smoothing assumptions and bounded support assumptions are applied this approximation is correct, this does not directly imply anything about the efficiency of QV. The reason is that ϵ may vanish at a rate faster than the approximation $v(u) \approx \epsilon u$ comes to hold. Establishing efficiency requires bounding the relative rates of these two limiting phenomena and not just showing that v is approximately linear for large N.

To see more clearly the role of our smoothness and bounded support assumptions, consider now the case when $\mu \neq 0$ and again suppose that $v(u) = \epsilon u$. Then V_{-1} would be normal with mean $2(N-1)\epsilon\mu$ and variance $4(N-1)\epsilon^2$ Thus by Equation 4 $v(u)$ would have to satisfy, by the definition of the normal density function,

$$v(u) = \frac{e^{-\frac{[v(u)+2(N-1)\epsilon\mu]^2}{8(N-1)\epsilon^2}}}{\sqrt{2\pi(N-1)}\epsilon} u.$$

Thus unless $v(u)$ is on the order of $2(N-1)\epsilon\mu$ (the total value of V_{-1}), the incentive to vote will die *exponentially*. It cannot be an equilibrium, however, for the votes of all voters to die exponentially, as in this case V_{-1} will with high probability be exponentially small

[25] To be more precise, GZ consider a version of the model with a different set of normalizations: the cost of buying votes is detail-dependent and equal to $\frac{1}{\sqrt{2\pi N}\sigma}$ where σ is the standard deviation of the normal distribution which where we have normalized to unity. As can easily be shown using a variation of our calculations below, the "natural" constant of proportionality ϵ with this cost function is 1 (actually $\sqrt{\frac{N}{N-1}}$, but close to 1 in the limit) and thus this conjecture is equivalent to GZ's statement in their Proposition 2, "For large electorates, truthful bidding constitutes a Bayes-Nash equilibrium..."

and thus any individual with a utility of sign opposite to μ will be able to guarantee victory with an exponentially small number of votes. This suggests that if most individuals purchase votes in an even approximately linear manner, there *must* be some individuals who have $v(u)$ on the order of $2(N-1)\epsilon\mu$ (viz. they buy as many votes as the rest of the population combined).

These extremely influential individuals could, therefore, disrupt the behavior of the distribution of V_{-1}. This suggests that characterizing the distribution of V_{-1}, or even basic smoothness properties of it, from first principles may be challenging. Without further analysis or restrictions, it would thus be difficult to apply the calculus methods that we have thus far relied upon to analyze QV. This is why we impose smoothing and boundedness: they provide sufficient restrictions on the behavior of votes and marginal pivotalities to provide a basis for bootstrapping towards the approximations upon which the "natural" smooth analysis of QV relies. While we believe that it is possible to establish our asymptotic optimality results without these restrictions, we have not been able to prove this rigorously thus far. See Weyl (2015) for some efforts in this direction.

5 Main Results

Our main results concern the structure of equilibria in the game described in the previous section when the number N of agents is large, and the implications for the welfare of QV.

5.1 Characterization of equilibrium in the zero mean case

The structure of a type-symmetric Bayes-Nash equilibrium differs radically depending on whether $\mu = 0$ or $\mu \neq 0$. The case of $\mu = 0$, although non-generic, is of particular interest because in some elections, for instance, when two candidates are vying for an elected office, the alternatives may be tailored so that an approximate population balance is achieved (Ledyard, 1984).

Theorem 1. For any sampling distribution F with mean $\mu = 0$ that satisfies the hypotheses above, constants $\epsilon_N \to 0$ exist such that in any type-symmetric Bayes-Nash equilibrium, $v(u)$ is C^∞ on $[\underline{u}, \overline{u}]$ and satisfies the following approximate proportionality rule:

$$\left|\frac{v(u)}{p_N u} - 1\right| \leq \epsilon_N \quad \text{where} \quad p_N = \frac{1}{2^{3/4}\sqrt{\sigma}\sqrt[4]{\pi(N-1)}}. \tag{5}$$

Furthermore, constants $\alpha_N, \beta_N \to 0$ exist such that in any equilibrium the vote total V

and expected welfare loss satisfy

$$|\mathbb{E}[V]| \leq \alpha_N \sqrt{\text{var}(V)} \quad \text{and} \quad WL < \beta_N. \tag{6}$$

Thus, in any equilibrium, agents buy votes approximately in proportion to their values u_i as described in Subsection 2.1. This *approximate proportionality rule* holds because in any equilibrium, each voter perceives approximately the same *marginal pivotality*, that is, roughly, the probability that the vote total V will be in the range $[-\delta, \delta]$, where a small increment to one's vote would affect the utility.

Given approximate proportionality, understanding why the number of votes a typical voter buys should be of order $N^{-1/4}$ is not difficult. If the vote function $v(u)$ in a Bayes-Nash equilibrium follows a proportionality rule $v(u) \approx \beta u$, the constant β must be the consensus marginal pivotality. On the other hand, by the local limit theorem of probability (see Feller (1971), ch. XVI), if $\beta = CN^{-\alpha}$ for some constants $C \neq 0$ and $\alpha \in \mathbb{R}$, the chance that $V \in [-\delta, \delta]$ would be of order $N^{\alpha - \frac{1}{2}}$, and so α must be $1/4$.

Although the relation (5) asserts the ratio $v(u)/u$ is approximately constant, our analysis will show it is not *exactly* constant: different agents will perceive slightly different marginal pivotalities. Consequently, the vote function $v(u)$ is a genuinely *nonlinear* function of u, as suggested in the previous section, and so even though $\mathbb{E}[U] = 0$, it need not be the case that $\mathbb{E}[V] = 0$. In particular, if $\mathbb{E}[V] \approx 0$ individuals with very large values will typically buy fewer votes in proportion to their values than those with smaller values, as their votes directly reduce the chance that $V \in [-\delta, \delta]$ by breaking the approximate aggregate tie.

Thus, to establish convergence, we must prove assertion (6), namely, that these non-linearities vanish rapidly enough that the bias created by non-linearity is smaller than the sampling variation in u. We demonstrate these bounds on convergence rates by using the *Edgeworth expansion* of the distribution of V_{-1}. Were $\mathbb{E}[V] = 0$ and the distribution of V_{-1} literally normal, a standard Taylor expansion and the $N^{-1/4}$–decay of $v(u)/u$ could be used directly to show that non-linearities vanish with N^{-1} even relative to the leading term of $v(u)/u$. A detailed application of this argument leads us to conjecture that, under the hypotheses of Theorem 1, the welfare loss of QV decays like $\mu_3^2/(16\sigma^6 N)$.[26] However, given that $\mathbb{E}[V]$ is not exactly 0, nor is V_{-1} literally normal, the arguments we use to establish (6) are subtler and consequently weaker.[27]

[26] See Weyl (2015) for relevant calculations.

[27] Our result implies that welfare loss tends towards zero compared to its maximum magnitude as would occur if the wrong decision were always made, and relative to the magnitude that would occur if a random choice were made. It thus dominates any decision that could be made with only the information available

5.2 Characterization of equilibrium in the non-zero mean case

When μ is not zero, the nature of equilibrium can be quite different: in particular, if the payoff function is sufficiently sharp (the support of its derivative is sufficiently small) then for sufficiently large N, any type-symmetric Bayes-Nash equilibrium has a large discontinuity in the extreme tail of the sampling distribution. Nevertheless, in all cases QV is asymptotically efficient, as the following theorem shows.

Theorem 2. Assume that the sampling distribution F has mean $\mu > 0$ and that F and Ψ satisfy the hypotheses above. Then there exist constants $\beta_N \to 0$ such that in any type-symmetric Bayes-Nash equilibrium $v(u)$,

$$WL < \beta_N. \tag{7}$$

Furthermore, there is a constant $\alpha \geq \delta$ depending on the sampling distribution F and the payoff function Ψ but not on N such that in any equilibrium $v(u)$, for any $\epsilon > 0$,

$$\sup_{\underline{u}+1/N \leq u \leq \overline{u}} |v(u) - \alpha\mu^{-1}u/N| < \alpha_N/N \quad \text{and hence} \tag{8}$$

$$P\{|V_N - \alpha| > \epsilon\} \leq \epsilon_N, \tag{9}$$

where $\epsilon_N, \alpha_N \to 0$ are constants that depend only on the sample size N, and not on the particular equilibrium.

This theorem allows for two cases. In the first, α, the asymptotic vote total, is equal to δ and thus the vote total is near δ with high probability for large N. This case occurs for large δ and thus relatively smooth payoff functions. In the second, $\alpha > \delta$, so that with high probability the vote total is outside $[-\delta, \delta]$ for large N. This case occurs for small δ and thus payoff functions that are sufficiently sharp (viz. sufficiently close to deterministic majority rule in purchased votes).

To see how this dichotomy arises, suppose that for some $\alpha \geq \delta$ there were a value $w \in (-\delta, 0)$ such that

$$(1 - \Psi(w))\,|\underline{u}| > (\alpha - w)^2; \tag{10}$$

then an agent with value u_i near the lower extreme $\underline{u}$ with knowledge that the one-out vote total $V_{-i} = \sum_{j \neq i} v_j$ is near α would find it worthwhile to buy $-\alpha + w$ votes and thus single-handedly move the vote total to w. Consequently, there can be no equilibrium in

to players ex-ante, unlike in the case when $\mu \neq 0$ and making the decision in the direction of μ (if feasible) achieves most potential welfare asymptotically.

which V_{-i} concentrates strictly below α if such a w exists, as this would lead a large number of individuals to wish to act in this extreme manner, contradicting the conjectured concentration. Therefore, in any equilibrium the voters with positive values u_j must buy enough votes to guarantee that the vote total concentrates at or above α. The minimal value $\alpha \geq \delta$ at which the advantage of radical behavior in the extreme lower tail disappears thus determines the equilibrium behavior (8). This will be at $\alpha = \delta$ *unless* there is a solution to the following problem.

Optimization Problem. There exists a unique $\alpha > \delta$ and a matching real number $w \in [-\delta, 0]$ such that

$$(1 - \Psi(w))\,|\underline{u}| = (\alpha - w)^2 \quad \text{and} \qquad (11)$$
$$(1 - \Psi(w'))\,|\underline{u}| \leq (\alpha - w')^2 \quad \text{for all } w' \neq w.$$

Proposition 1. If $\delta < 1/\sqrt{2}$ then there exists a unique $\alpha > \delta$ and a matching real number $w \in [-\delta, 0]$ satisfying the Optimization Problem (11).

Thus as long as δ is sufficiently small and thus the payoff sufficient close to sharp majority rule, the Optimization Problem has a unique solution. We thus focus on this latter case, as we view our smoothing function Ψ as a technical simplification rather than an substantive assumption and prefer to focus on cases close to sharp majority rule. When this is true, type-symmetric Bayes-Nash equilibria take a rather interesting form in which extremists must appear, with vanishing probability, as the following theorem shows.

Theorem 3. Assume that the sampling distribution F has mean $\mu > 0$ and that F and Ψ satisfy the hypotheses above. Assume further that the Optimization Problem (11) has a unique solution (α, w). Then exists a constant $\zeta > 0$ depending on F such that for any $\epsilon > 0$ and any type-symmetric Bayes-Nash equilibrium $v(u)$, when N is sufficiently large,

(i) $v(u)$ has a single discontinuity at u_*, where $|u_* + |\underline{u}| - \zeta N^{-2}| < \epsilon N^{-2}$;

(ii) $|v(u) + \alpha - w| < \epsilon$ for $u \in [\underline{u}, u_*)$; and

(iii) the approximate proportionality rule (8) holds for all $u \in [u_*, \overline{u}]$.

Theorem 3 implies that an agent with value u will buy approximately $\alpha\mu^{-1}u/N$ votes unless u is in the extreme lower tail of F. Since such an exceptional agent occurs in the full sample only with probability $\approx \zeta N^{-1} f(\underline{u})$, it follows by the law of large numbers that with probability $\approx 1 - \zeta N^{-1} f(\underline{u})$, the vote total will be very near α. If, on the other hand, the sample contains an agent with value less than u_* then this agent will buy approximately $\alpha - w \approx -\sqrt{|\underline{u}|}$ votes, enough to move the overall vote total close to w. Agents of the first type will be called *moderates*, and agents of the second kind *extreme contrarians* or *extremists* for short. Because the tail region in which extremists reside has $F-$probability on the order N^{-2}, the sample of agents will contain an extremist with probability only on the order N^{-1}, and will contain two or more extremists with probability on the order N^{-2}. Given that the sample contains no extremists, the conditional probability that $|V - \alpha| > \epsilon$ is $O(e^{-\varrho n})$ for some $\varrho > 0$, by standard large deviations estimates, and so the event that $V < 0$ essentially coincides with the event that the sample contains an extremist.

Why does equilibrium take the somewhat counter-intuitive form described in Theorem 3? An agent i with value u_i in the "bulk" of the sampling distribution F, there is very little information about the vote total V in the agent's value u_i, and so for most such agents the marginal pivotality $\mathbb{E}\left[\psi\left(V_{-i} + v(u_i)\right)\right]$ will be approximately $\mathbb{E}\left[\psi(V)\right]$. This implies that in the bulk of the distribution the function $v(u)$ will be approximately linear in u. Therefore, by the law of large numbers, the vote total will, with high probability, be near $N\mathbb{E}\left[\psi(V)/2\mu\right]$.

Because $\mu > 0$, agents with negative values will, with high probability, be on the losing side of the election. However, if $\mathbb{E}\left[\psi(V)\right]$ were small enough that $N\mathbb{E}\left[\psi(V)/2\mu\right]$ decayed with N, agents with negative values could overcome the votes of all other individuals at sufficiently small cost that all would eventually wish to be extremists. Such a high level of extremist activity in turn would imply $\mathbb{E}\left[\psi(V)\right]$ is large, because any time an extremist exists the conditional expectation $\psi(V)$ must be large enough to satisfy her first-order condition, which involves buying many votes. Consequently, $N\mathbb{E}\left[\psi(V)\mu\right]$ must remain bounded away from 0.

On the other hand, if $N\mathbb{E}\left[\psi(V)\right]$ grows unboundedly with N, no individual could profitably act as an extremist, so no extremist will exist and $\mathbb{E}\left[\psi(V)\right]$ would be exponentially small, which is impossible. Thus, the aggregate number of votes must concentrate near a constant value, and so most voters must buy on the order of $1/N$ votes. For this scenario to occur, $\mathbb{E}\left[\psi(V)\right]$ must decay as $1/N$. But the primary contribution to this expectation must come from the event in which an extremist exists, and so the probability of this event must decay as $1/N$. Because it is only in this event that welfare loss can occur, we have the following corollary which provides a rate of decay of welfare loss in the generic $\mu \neq 0$ case.

Corollary 1. Under the hypotheses of Theorem 3, WL is of order $O(N^{-1})$.

5.3 Some Remarks on the Proofs

The heuristic arguments of the preceding two subsections rely on central limit approximations and large deviations estimates for the aggregate vote total. Applying these theorems is not justified *a priori*, however, because despite the fact that the distribution of values is exogeneous, the distribution of votes is endogenous. Moreover, precisely because marginal pivotality is not constant, the distribution of votes may be different from that of values in important respects. This endogeneity severely circumscribes our ex-ante knowledge about how marginal pivotality itself behaves. Thus, we will be forced to squeeze out information about the vote function $v(u)$ by a bootstrapping procedure, at each step using new information about $v(u)$ to make a more precise approximation of the marginal pivotality and thereby obtain even more precise information about the vote function $v(u)$. The key steps, which are established formally in our online appendix, are as follows.

(1) *Weak Consensus:* The most crucial step is to show that *most* agents (all but those whose values u are in the tails of the distribution F) will have similar assessments of the conditional distribution (given their knowledge of their own values) of the vote total, and hence the marginal pivotalities. The key to this *weak consensus* principle is a basic result in the theory of random sampling: the probability of obtaining exactly k individuals in a random sample with values in a given interval J is virtually the same as the probability of obtaining exactly $k+1$ such individuals *unless* J is so small or so far out in the tail of the sampling distribution that the likelihood of obtaining more than a very small number of individuals with values in J is negligible. These notions of "smallness" are quantified in Lemma 5 of our online appendix; it implies for any two individuals with values u_i, u_j in the bulk of the distribution F, the ratio $\frac{v(u_i)/u_i}{v(u_j)/u_j}$ is bounded above and below.

(2) *Concentration:* We then employ a *concentration inequality* for sums of i.i.d. random variables that bounds the probability that such a sum will fall in a given interval. This bound, involving the variance, third moment, and N, will lead to bounds on the size of $v(u)$. In a nutshell, the argument is as follows.

For an agent with value u, the marginal pivotality is the expectation $\mathbb{E}\left[\psi(V_{-1}+v(u))\right]$. Because ψ is non-zero only in the interval $(-\delta, \delta)$, the expectation will be large only if the distribution of $V_{-1}+v(u)$ puts a substantial mass on this interval. But the concentration inequality implies the distribution of the sum V_{-1} will be highly concentrated just if the individual summands $v(u_j)$ have small variance, which will be the case just when they are (mostly) small. By the necessary condition for an equilibrium, this can occur only if the marginal pivotalities $\mathbb{E}\left[\psi(V_{-j}+v(u_j))\right]$ are (mostly) small. Therefore, by the weak consensus principle, if $v(u)/u$ is even moderately large, the value u must be in one of the extreme tails of the distribution F, because for nearly all other values u the marginal pivotality will be small.

A careful rendition of this argument will show that (i) extremists must have values u within distance $O(N^{-3/2})$ of one of the endpoints $\underline{u}, \overline{u}$ of the support interval, and (ii) the number of votes $|v(u)|$ that any agent in the bulk of the distribution buys cannot be larger than $O(N^{-1/4})$.

(3) *Discontinuities and Smoothness:* At any discontinuity of the vote function $v(u)$, two distinct solutions (the right and left limits $v(u+)$ and $v(u-)$) of the necessary condition (3) must exist. Because ψ is smooth, the derivative ψ' exists and is continuous everywhere, and so the mean value theorem implies that at any such discontinuity, some $\tilde{v} \in [v(u-), v(u+)]$ must exist at which $\mathbb{E}\left[\psi'(\tilde{v} + V_{-1})u\right] = 2$. But because ψ' is non-zero only in the interval $(-\delta, \delta)$, the existence of $\tilde{v}$ implies, once again, that the distribution of V_{-1} is highly concentrated; thus, a variation of the argument in (2) implies that (i) the size of any discontinuity must be bounded below, and (ii) discontinuities can occur only at values u in the extreme tails of F.

Because the vote function $v(u)$ is monotone in any equilibrium, it must be differentiable almost everywhere. At any u where the derivative $v'(u)$ exists, the necessary condition (3) can be differentiated, yielding the identity

$$v'(u) = \frac{\mathbb{E}\left[\psi(v(u) + V_{-1})\right]}{2 - \mathbb{E}\left[\psi'(v(u) + V_{-1})\right] u}.$$

The quantity on the right side varies continuously with u in any interval where $v(u)$ is continuous and $E\psi'(v(u) + V_{-1})u < 2$, and hence v' extends continuously to any such interval. Thus we conclude, by another use of the concentration inequalities, that $v(u)$ is not only continuous but continuously differentiable at all u except in the extreme tails of F.

(4) *Approximate Proportionality:* Weak consensus tells us the ratio $v(u)/u$ does not vary, in relative terms, by more than a bounded amount in the bulk of the distribution F, but to complete the proofs we will need something stronger: that when N is large, the ratio $v(u)/u$ is nearly constant except in the tails of F. To establish this result, we bring the various bits of information gleaned from the analysis in steps 1-3 to bear on the necessary condition (3). Because ψ is smooth and has compact support, it and all of its derivatives are uniformly continuous and uniformly bounded, and so the function $v \mapsto E\psi(v + V_{-1})$ is differentiable, with derivative $E\psi'(v+V_{-1})$. Consequently, by an application of Taylor's

theorem to the identity (3), a $\tilde{v}(u)$ intermediate between 0 and $v(u)$ exists such that

$$2v(u) = \mathbb{E}\left[\psi(V_{-1})u\right] + \mathbb{E}\left[\psi'(\tilde{v}(u) + V_{-1})v(u)u\right].$$

Concentration implies that, for u in the bulk of the distribution F, when N is large, the expectation $\mathbb{E}\left[\psi'(\tilde{v}(u) + V_{-1})\right]$ is small; thus, the *approximate proportionality* rule $2v(u) \approx \mathbb{E}\left[\psi(V_{-1})u\right]$ must hold except in the tails of F. A more delicate analysis, using the smoothness of v (Step 3), will show the approximate proportionality rule extends to all but the *extreme* tails, that is to all u not within distance $N^{-3/2}$ of one of the endpoints $\underline{u}, \overline{u}$.

The upshot of approximate proportionality is that the contribution to the one-out vote total V_{-1} of those terms $v(u_i)$ for which u_i is not in the extreme tails of F can be bounded above and below by scalar multiples of the sums of the corresponding values u_i. Since these values are drawn i.i.d. from a distribution with bounded support, standard results of probability theory (central limit theorem, Berry-Esseen bounds, Hoeffding's inequality) now apply to the distribution of V_{-1}.

(5) *Non-zero mean case*: In the non-zero mean case, we first rule out discontinuities near the upper endpoint $\overline{u}$ by arguing that, unless the sample contains an extremist with value near $\underline{u}$, an event of vanishingly small probability, the conditional probability that the sum of the moderate votes will exceed δ is exponentially close to 1, so voters with values near $\overline{u}$ need not buy more than $O(N^{-1})$ votes to ensure their side wins the election with all but vanishing probability. From there we closely follow the heuristic argument of the previous subsection.

(6) *Zero-mean case*: This case is technically more delicate because the Edgeworth expansion discussed above must be conducted not about 0 as supposed above, but rather about the still unknown $\mathbb{E}[V]$. Thus, another bootstrapping argument is needed to deduce that $\mathbb{E}[V]$ is of a smaller order of magnitude than the standard deviation of V. Given this bound, it then follows that the lead term in the Edgeworth series (the usual normal approximation) dominates, and the results claimed in Theorem 1 then follow.

6 Conclusion

This paper makes three contributions. First, we propose a mechanism for the binary collective decision problem, Quadratic Voting (QV). Second, we argue heuristically and by drawing on the literature that this mechanism is more natural than existing efficient mechanisms. Finally, we develop tools to overcome these challenges, allowing us to fully characterize all symmetric large population equilibria and thus prove convergence to optimality in the most canonical private information model.

For our results to be of direct practical relevance, however, analysis must go much farther. Formal analysis of QV's robustness to collusion, aggregate uncertainty and voter behavior, extending the approximation-based results of Kaplow and Kominers (Forthcoming) and Weyl (Forthcoming) is important to investigate the informal discussion of Subsection 2.2. Allowing for partially "common values" in which voters have information that is relevant to the choices of other voters is an other important direction. QV's performance in such an environment is of great interest, as it seems plausible it could allow for the expression of the quality of information held by different individuals and thus improve on the information aggregation that occurs in large standard voting economies (Feddersen and Pesendorfer, 1997).

Furthermore, the version of QV we presented here is extremely specific; many applications will require a more flexible mechanism. Proving extensions of our results to versions of QV where an artificial currency to trade off different decisions without transfers and thus Pareto efficiency is the appropriate criterion will be critical in many settings where equity is a leading concern. In many contexts there are more than two options available; determining whether QV can help overcome some of the paradoxes in those settings (Arrow, 1951; Gibbard, 1973; Satterthwaite, 1975) by incorporating cardinal information and weakening strategy-proofness is an exciting theoretical problem. In other contexts choices will lie in a continuous space as in Groves and Ledyard (1977a) and HZ; extending our large population convergence results to that setting and deriving an algorithm that under truthful reporting is guaranteed to converge to an equilibrium are important for ensuring such an approach could be practically applied.

Finally, theoretical economic analysis alone is both unlikely to produce persuasive evidence for the value of QV nor to allow analysts to design practical implementations that are likely to work well in a modern society. One important issue is user interface, given that quadratic functions are not widely understood by the public. Quarfoot et al. (Forthcoming) show one solution to this problem, but there is great room for improvement. Another issue relates to security and checks on collusion and fraud. Park and Rivest (Forthcoming) propose both physical and electronic security protocols that could help both deter collusion and fraud and ensure that such behavior could easily be detected by authorities. More broadly, a range of sociological, psychological, technical and infrastructure issues would need to be addressed before QV could be adopted on a wide scale. Even without such adoption, QV may offer an interesting benchmark model of the incentives necessary to achieve optimality in politics against which current institutions could be compared.

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